Explicit and Exact Nontravelling Wave Solutions of Konopelchenko-Dubrovsky Equations

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Based on a new first-order nonlinear ordinary differential equation with a sixth-degree nonlinear term and some of its special solutions, a generalized transformation method is proposed to obtain more general exact solutions of the (2+1)-dimensional Konopelchenko-Dubrovsky equations. As a result, new exact nontravelling wave solutions are obtained including soliton-like solutions and trigonometric function solutions, from which all travelling wave solutions obtained by B. C. Li and Y. F. Zhang [Chaos, Solitons and Fractals (2007), doi:10.1016/j.chaos.2007.01.059] can be recovered as special cases. Compared with Li and Zhang's method and the method of D. J. Huang, D. S. Li, and H. Q. Zhang [Commun. Theor. Phys. (Beijing, China) **44**, 969 (2005)], D. J. Huang and H. Q. Zhang [Rep. Math. Phys. **57**, 257 (2006)], D. J. Huang, D. S. Li, and H. Q. Zhang [Chaos, Solitons and Fractals **31**, 586 (2007)], the proposed method is more powerful in searching for exact solutions of nonlinear evolution equations in mathematical physics.

Key words: Nonlinear Evolution Equations; Nontravelling Wave Solutions; Soliton-Like Solutions; Trigonometric Function Solutions.

1. Introduction

The investigation of exact solutions of nonlinear evolution equations (NLEEs) plays an important role in the study of nonlinear physical phenomena. In the past several decades, many effective methods for obtaining exact solutions of NLEEs have been presented, such as the inverse scattering method [1], Hirota's bilinear method [2], Bäcklund transformation [3], Painlevé expansion [4], sine-cosine method [5], homogenous balance method [6], homotopy perturbation method [7 – 9], variational method [10 – 13], asymptotic methods [14], non-perturbative methods [15], Adomian decomposition method [16], tanh-function method [17 – 21], algebraic method [22 – 25], Jacobi elliptic function expansion method [26 – 28], F-expansion method [29 – 34], and auxiliary equation method [35 – 39].

With the development of computer science, recently, directly searching for exact solutions of NLEEs has attracted much attention. This is due to the availability of symbolic computation systems like *Mathematica* or *Maple* which enable us to perform the complex and tedious computation on computers. For example, the Exp-function method proposed by He and Wu [40] is a straightforward and concise method for obtaining generalized solitonary solutions, solitary wave solutions and periodic solutions of NLEEs. The solution procedure of this method, by the help of *Mathematica*, is of utter simplicity, and this method has been successfully applied to many kinds of NLEEs [41 – 45].

Recently, Huang et al. [46 – 48] proposed a new direct method by introducing a new first-order nonlinear ordinary differential equation with a sixth-degree nonlinear term and its known solutions to construct exact travelling wave solutions of NLEEs. Very recently, Li and Zhang [49] proposed a transformation method to develop this method.

In the present paper, we introduce a general transformation to generalize and improve the work made in [46 – 49] for finding more general exact solutions of NLEEs. In order to illustrate the effectiveness and convenience of the method, we would like to consider the (2+1)-dimensional Konopelchenko–Dubrovsky (KD) equations [50]

$$u_t - u_{xxx} - 6\beta u u_x + \frac{3}{2}\alpha^2 u^2 u_x - 3v_y + 3\alpha u_x v = 0, \quad (1)$$

$$u_y = v_x, \quad (2)$$

where α and β are real constants. For $u_y = 0$, (1) is the Gardner equation [combined Korteweg – de Vries

(KdV) and modified KdV equation]. For $\alpha = 0$, (1) is the well-known Kadomtsev-Petviashvili (KP) equation, and the modified KP equation reads from (1) for $\beta = 0$. Many authors [21, 31, 33, 51–55] have studied (1) and (2), and obtained many types of exact solutions. Very recently, Li and Zhang [49] found new travelling wave solutions by means of a transformation method. In this paper, we will obtain new and more general exact solutions namely nontravelling wave solutions, each of which can not be obtained by the methods described in [46–49], and besides, all solutions obtained in [49] can be recovered as special cases.

The rest of this paper is organized as follows: in Section 2, we give the description of a generalized transformation method; in Section 3, we use the generalized transformation method to obtain explicit and exact nontravelling wave solutions of the (2+1)-dimensional KD equations; in Section 4, some conclusions are given.

2. Description of the Generalized Transformation Method

For a given NLEE, with independent variables $X = (x, y, \ldots, t)$ and a dependent variable u,

$$F(u, u_t, u_x, u_y, \ldots, u_{xt}, u_{yt}, \ldots, u_{tt}, u_{xx}, u_{yy}, \ldots) = 0, \quad (3)$$

we seek its solutions in the more general form [29]:

$$u = a_0(X) + \sum_{i=1}^{n} \{a_i(X)\phi^i(\xi) + b_i(X)\phi^{-i}(\xi) + c_i(X)\phi^{i-1}(\xi)\phi'(\xi) + d_i(X)\phi^{-i}(\xi)\phi'(\xi)\}, \quad (4)$$

where $a_i(X)$, $b_i(X)$, $c_i(X)$, $d_i(X)$ and $\xi = \xi(X)$ are all functions to be determined later. $\phi(\xi)$ satisfies the auxiliary ordinary differential equation

$$\phi'^2(\xi) = h_0 + h_1\phi(\xi) + h_2\phi^2(\xi) + h_3\phi^3(\xi) + h_4\phi^4(\xi) + h_5\phi^5(\xi) + h_6\phi^6(\xi), \quad (5)$$

where h_j $(j = 0, 1, 2, \ldots, 6)$ are all parameters; the prime denotes $\mathrm{d}/\mathrm{d}\xi$.

It can be easily found that the transformation (4) is more general than those constructed in [46–49]. To be more precise, if $b_i(X) = c_i(X) = d_i(X) = 0$, $a_0(X) = a_0$ and $a_i(X) = a_i$ are constants, and ξ is merely a linear function of x and t, namely $\xi = x - ct + l$, then (4) exactly becomes the one used in [46–48]:

$$u = \sum_{i=0}^{n} a_i\phi^i(\xi).$$

If $c_i(X) = d_i(X) = 0$ and $\xi = k(x + ly - \lambda t)$, then (4) reduces to

$$u = \sum_{i=0}^{n} a_i\phi^i(\xi) + \sum_{i=1}^{n} + b_i\phi^{-i}(\xi),$$

which is equivalent to that proposed in [49].

It shows that taking full advantages of the transformation (4) we may obtain new and more general exact solutions of NLEEs including not only all travelling wave solutions obtained by the methods of [46–49] but also new types of exact solutions which can not be obtained by the method of [46–49]. In order to determine u explicitly, we take the following four steps:

Step 1. Determine the integer n. Substituting (4) along with (5) into (3) and balancing the highest-order partial derivative with the nonlinear term(s) in (3), we then obtain the value of n.

Step 2. Derive a system of equations. Substituting (4), given the value of n obtained in Step 1, along with (5) into (3), collecting coefficients of $\phi^j(\xi)\phi'^l(\xi)$ $(l = 0, 1; j = 0, \pm 1, \pm 2, \ldots)$, then setting each coefficient to zero, we can derive a set of over-determined partial differential equations for $a_0(X)$, $a_i(X)$, $b_i(X)$, $c_i(X)$, $d_i(X)$ and $\xi(X)$.

Step 3. Solve the system of equations. Solving the system of over-determined partial differential equations derived in Step 2 by use of *Mathematica*, we obtain the explicit expressions for $a_0(X)$, $a_i(X)$, $b_i(X)$, $c_i(X)$, $d_i(X)$ and ξ.

Step 4. Obtain exact solutions. By using the results obtained in the above steps, we can derive a series of fundamental solutions of (3) depending on the solution $\phi(\xi)$ of (5). Given different values of h_j $(j = 0, 1, 2 \ldots, 6)$, (5) has many kinds of special solutions. Some of them are listed in [23] under the condition $h_5 = h_6 = 0$. We list and consider in this paper only the ones with $h_6 \neq 0$ as follows:

I. Suppose that $h_0 = h_1 = h_3 = h_5 = 0$, $h_6 < 0$ and $h_4^2 - 4h_2h_6 > 0$.

(i) If $h_2 > 0$, $h_4 < 0$, then (5) has the following solutions:

$$\phi(\xi) = \left\{ \frac{2h_2 \operatorname{sech}^2\left(\sqrt{h_2}(\xi + \xi_0)\right)}{2\sqrt{h_4^2 - 4h_2h_6} - \left(\sqrt{h_4^2 - 4h_2h_6} + h_4\right) \operatorname{sech}^2(\sqrt{h_2}(\xi + \xi_0))} \right\}^{\frac{1}{2}}, \quad (6)$$

$$\phi(\xi)=\left\{\frac{2h_2\operatorname{csch}^2\left(\pm\sqrt{h_2}(\xi+\xi_0)\right)}{2\sqrt{h_4^2-4h_2h_6}+\left(\sqrt{h_4^2-4h_2h_6}-h_4\right)\operatorname{csch}^2\left(\pm\sqrt{h_2}(\xi+\xi_0)\right)}\right\}^{\frac{1}{2}}. \tag{7}$$

(ii) If $h_2<0$, $h_4\geq 0$, then (5) has the following solutions:

$$\phi(\xi)=\left\{\frac{-2h_2\sec^2\left(\sqrt{-h_2}(\xi+\xi_0)\right)}{2\sqrt{h_4^2-4h_2h_6}-\left(\sqrt{h_4^2-4h_2h_6}-h_4\right)\sec^2\left(\sqrt{-h_2}(\xi+\xi_0)\right)}\right\}^{\frac{1}{2}}, \tag{8}$$

$$\phi(\xi)=\left\{\frac{2h_2\csc^2\left(\pm\sqrt{-h_2}(\xi+\xi_0)\right)}{2\sqrt{h_4^2-4h_2h_6}-\left(\sqrt{h_4^2-4h_2h_6}+h_4\right)\csc^2\left(\pm\sqrt{-h_2}(\xi+\xi_0)\right)}\right\}^{\frac{1}{2}}. \tag{9}$$

II. Suppose that $h_1=h_3=h_5=0$, $h_0=\frac{8h_2^2}{27h_4}$ and $h_6=\frac{h_4^2}{4h_2}$.

(i) If $h_2<0$ and $h_4>0$, then (5) has the following solutions:

$$\phi(\xi)=\left\{-\frac{8h_2\tanh^2\left(\pm\sqrt{-\frac{h_2}{3}}(\xi+\xi_0)\right)}{3h_4\left[3+\tanh^2\left(\pm\sqrt{-\frac{h_2}{3}}(\xi+\xi_0)\right)\right]}\right\}^{\frac{1}{2}}, \tag{10}$$

$$\phi(\xi)=\left\{-\frac{8h_2\coth^2\left(\pm\sqrt{-\frac{h_2}{3}}(\xi+\xi_0)\right)}{3h_4\left[3+\coth^2\left(\pm\sqrt{-\frac{h_2}{3}}(\xi+\xi_0)\right)\right]}\right\}^{\frac{1}{2}}. \tag{11}$$

(ii) If $h_2>0$ and $h_4<0$, then (5) has the following solutions:

$$\phi(\xi)=\left\{\frac{8h_2\tan^2\left(\pm\sqrt{\frac{h_2}{3}}(\xi+\xi_0)\right)}{3h_4\left[3-\tan^2\left(\pm\sqrt{\frac{h_2}{3}}(\xi+\xi_0)\right)\right]}\right\}^{\frac{1}{2}}, \tag{12}$$

$$\phi(\xi)=\left\{\frac{8h_2\cot^2\left(\pm\sqrt{\frac{h_2}{3}}(\xi+\xi_0)\right)}{3h_4\left[3-\cot^2\left(\pm\sqrt{\frac{h_2}{3}}(\xi+\xi_0)\right)\right]}\right\}^{\frac{1}{2}}. \tag{13}$$

III. Suppose that $h_0=h_1=h_3=h_5=0$ and $h_6=\frac{h_4^2}{4h_2}$.

If $h_2>0$, $h_4<0$, then (5) has the following solutions:

$$\phi(\xi)=\left\{-\frac{h_2}{h_4}\left[1+\tanh\left(\pm\sqrt{h_2}(\xi+\xi_0)\right)\right]\right\}^{\frac{1}{2}}, \tag{14}$$

$$\phi(\xi)=\left\{-\frac{h_2}{h_4}\left[1+\coth\left(\sqrt{h_2}(\xi+\xi_0)\right)\right]\right\}^{\frac{1}{2}}. \tag{15}$$

Remark 1. In order to determine the explicit solutions of the partial differential equations derived in Step 2, we may choose special forms of $a_0(X)$, $a_i(X)$, $b_i(X)$, $c_i(X)$, $d_i(X)$ and $\xi(X)$ as we do in Section 3.

3. Nontravelling Wave Solutions of the (2+1)-Dimensional KD Equations

In this section, let us consider the (2+1)-dimensional KD equations (1) and (2). According to Step 1, we get $n=2$ for u and v. In order to search for explicit solutions, we suppose that (1) and (2) have the formal solutions

$$\begin{aligned}u=&\,a_0(y,t)+a_1(y,t)\phi(\xi)+a_2(y,t)\phi^2(\xi)\\&+b_1(y,t)\phi^{-1}(\xi)+b_2(y,t)\phi^{-2}(\xi)\\&+c_1(y,t)\phi'(\xi)+c_2(y,t)\phi'(\xi)\phi(\xi)\\&+d_1(y,t)\phi'(\xi)\varphi^{-1}(\xi)+d_2(y,t)\phi'(\xi)\phi^{-2}(\xi),\end{aligned} \tag{16}$$

$$\begin{aligned}v=&\,A_0(y,t)+A_1(y,t)\phi(\xi)+A_2(y,t)\phi^2(\xi)\\&+B_1(y,t)\phi^{-1}(\xi)+B_2(y,t)\phi^{-2}(\xi)\\&+C_1(y,t)\phi'(\xi)+C_2(y,t)\phi'(\xi)\phi(\xi)\\&+D_1(y,t)\phi'(\xi)\phi^{-1}(\xi)+D_2(y,t)\phi'(\xi)\phi^{-2}(\xi),\end{aligned} \tag{17}$$

where $\xi = kx + \eta(y,t)$ and k is a nonzero constant.

Substituting (16) and (17) along with (5) into (1) and (2), the left-hand sides of (1) and (2) are converted into two polynomials of $\phi'^i(\xi)\phi^j(\xi)$ ($i = 0,1; j = 0,\pm1,\pm2,\ldots$). Then setting each coefficient to zero, we get a set of over-determined partial differential equations for $a_0(y,t)$, $a_1(y,t)$, $a_2(y,t)$, $b_1(y,t)$, $b_2(y,t)$, $c_1(y,t)$, $c_2(y,t)$, $d_1(y,t)$, $d_2(y,t)$, $A_0(y,t)$, $A_1(y,t)$, $A_2(y,t)$, $B_1(y,t)$, $B_2(y,t)$, $C_1(y,t)$, $C_2(y,t)$, $D_1(y,t)$, $D_2(y,t)$ and $\eta(y,t)$. Solving the set of over-determined partial differential equations by use of *Mathematica*, we get the following results:

Case 1.

$$a_0(y,t) = \frac{kp_1\alpha - f(t)}{k\alpha}, \quad a_1(y,t) = a_1, \quad a_2(y,t) = a_2, \quad b_1(y,t) = 0, \tag{18}$$

$$b_2(y,t) = 0, \quad c_1(y,t) = 0, \quad c_2(y,t) = 0, \quad d_1(y,t) = 0, \quad d_2(y,t) = 0, \tag{19}$$

$$A_0(y,t) = \left[p_2 + 6a_2^2k(p_1\alpha^2 - 2\beta)f(t) + 3a_2^2\alpha f^2(t) - 2a_2^2k\alpha(yf'(t) + g'(t))\right]\left[6a_2^2k^2\alpha^2\right]^{-1}, \tag{20}$$

$$A_1(y,t) = \frac{a_1 f(t)}{k}, \quad A_2(y,t) = \frac{a_2 f(t)}{k}, \quad B_1(y,t) = 0, \quad B_2(y,t) = 0, \quad C_1(y,t) = 0, \tag{21}$$

$$C_2(y,t) = 0, \quad D_1(y,t) = 0, \quad D_2(y,t) = 0, \quad \eta(y,t) = yf(t) + g(t), \tag{22}$$

$$h_0 = h_0, \quad h_1 = a_1(32p_2 + k^2\alpha(a_1^4\alpha^2 + 96a_2^2p_1(p_1\alpha^2 - 4\beta) - 16a_1^2a_2(p_1\alpha^2 - 2\beta)))(256a_2^3k^4\alpha)^{-1}, \tag{23}$$

$$h_2 = \left[32p_2 - 3k^2\alpha(a_1^4\alpha^2 - 32a_2^2p_1(p_1\alpha^2 - 4\beta) - 16a_1^2a_2(p_1\alpha^2 - 2\beta))\right]\left[256a_2^2k^4\alpha\right]^{-1}, \tag{24}$$

$$h_3 = \frac{a_1(a_1^2\alpha^2 + 16a_2(p_1\alpha^2 - 2\beta))}{32a_2k^2}, \quad h_4 = \frac{11a_1^2\alpha^2 + 16a_2(p_1\alpha^2 - 2\beta)}{64k^2}, \tag{25}$$

$$h_5 = \frac{3a_1a_2\alpha^2}{16k^2}, \quad h_6 = \frac{a_2^2\alpha^2}{16k^2}, \tag{26}$$

where $f(t)$ and $g(t)$ are arbitrary functions of t, $f'(t) = \mathrm{d}f(t)/\mathrm{d}t$, $g'(t) = \mathrm{d}g(t)/\mathrm{d}t$, a_2 is a nonzero constant, while a_1, p_1 and p_2 are all arbitrary constants.

Case 2.

$$a_0(y,t) = \frac{2k\beta - \alpha f(t)}{k\alpha^2}, \quad a_1(y,t) = 0, \quad a_2(y,t) = \pm\frac{2k\sqrt{h_6}}{\alpha}, \quad b_1(y,t) = 0, \tag{27}$$

$$b_2(y,t) = 0, \quad c_1(y,t) = 0, \quad c_2(y,t) = 0, \quad d_1(y,t) = \pm\frac{2k}{\alpha}, \quad d_2(y,t) = 0, \tag{28}$$

$$A_0(y,t) = \left[12\beta^2k^2 - 4h_2k^4\alpha^2 + 3\alpha^2f^2(t) - 2k\alpha^2(yf'(t) + g'(t))\right]\left[6k^2\alpha^3\right]^{-1}, \tag{29}$$

$$A_1(y,t) = 0,$$

$$A_2(y,t) = \pm\frac{2\sqrt{h_6}f(t)}{\alpha}, \quad B_1(y,t) = 0, \quad B_2(y,t) = 0, \quad C_1(y,t) = 0, \tag{30}$$

$$C_2(y,t) = 0, \quad D_1(y,t) = \pm\frac{2f(t)}{\alpha}, \quad D_2(y,t) = 0, \quad \eta(y,t) = yf(t) + g(t), \tag{31}$$

$$h_0 = h_0, \quad h_1 = 0, \quad h_2 = h_2, \quad h_3 = 0, \quad h_4 = h_4, \quad h_5 = 0, \quad h_6 = h_6, \tag{32}$$

where $f(t)$ and $g(t)$ are arbitrary functions of t, $f'(t) = \mathrm{d}f(t)/\mathrm{d}t$, $g'(t) = \mathrm{d}g(t)/\mathrm{d}t$.

Now we use Case 1 to obtain exact solutions of (1) and (2). When setting

$$a_1 = 0, \quad h_0 = \frac{(p_2 + 3a_2^2k^2p_1\alpha(p_1\alpha^2 - 4\beta))^2}{54a_2^5k^6\alpha^2(p_1\alpha^2 - 2\beta)},$$
$$2a_2^2k^2(p_1\alpha^2 - 2\beta)^2 - \alpha(p_2 + 3a_2^2k^2p_1\alpha(p_1\alpha^2 - 4\beta)) = 0$$

in Case 1, we can verify that $h_1 = h_3 = h_5 = 0$, and h_0, h_2, h_4 and h_6 satisfy the conditions $h_0 = \frac{8h_2^2}{27h_4}$, $h_6 = \frac{h_4^2}{4h_2}$ in II.

If $(p_2 + 3a_2^2k^2p_1\alpha(p_1\alpha^2 - 4\beta))/\alpha < 0$ and $a_2(p_1\alpha^2 - 2\beta) > 0$, then from (10) and (11) we obtain soliton-like solutions of (1) and (2):

$$u = \frac{kp_1\alpha - f(t)}{k\alpha} - \frac{4(p_2 + 3a_2^2k^2p_1\alpha(p_1\alpha^2 - 4\beta))\tanh^2\left(\pm\sqrt{-\frac{p_2+3a_2^2k^2p_1\alpha(p_1\alpha^2-4\beta)}{24a_2^2k^4\alpha}}(\xi + \xi_0)\right)}{3a_2^2k^2\alpha(p_1\alpha^2 - 2\beta)\left[3 + \tanh^2\left(\pm\sqrt{-\frac{p_2+3a_2^2k^2p_1\alpha(p_1\alpha^2-4\beta)}{24a_2^2k^4\alpha}}(\xi + \xi_0)\right)\right]}, \tag{33}$$

$$v = \frac{p_2 + 6a_2^2k(p_1\alpha^2 - 2\beta)f(t) + 3a_2^2\alpha f^2(t) - 2a_2^2k\alpha(yf'(t) + g'(t))}{6a_2^2k^2\alpha^2}$$
$$- \frac{4(p_2 + 3a_2^2k^2p_1\alpha(p_1\alpha^2 - 4\beta))f(t)\tanh^2\left(\pm\sqrt{-\frac{p_2+3a_2^2k^2p_1\alpha(p_1\alpha^2-4\beta)}{24a_2^2k^4\alpha}}(\xi+\xi_0)\right)}{3a_2^2k^3\alpha(p_1\alpha^2-2\beta)\left[3+\tanh^2\left(\pm\sqrt{-\frac{p_2+3a_2^2k^2p_1\alpha(p_1\alpha^2-4\beta)}{24a_2^2k^4\alpha}}(\xi+\xi_0)\right)\right]}, \tag{34}$$

where $\xi = kx + yf(t) + g(t)$;

$$u = \frac{kp_1\alpha - f(t)}{k\alpha} - \frac{4(p_2 + 3a_2^2k^2p_1\alpha(p_1\alpha^2 - 4\beta))\coth^2\left(\pm\sqrt{-\frac{p_2+3a_2^2k^2p_1\alpha(p_1\alpha^2-4\beta)}{24a_2^2k^4\alpha}}(\xi+\xi_0)\right)}{3a_2^2k^2\alpha(p_1\alpha^2-2\beta)\left[3+\coth^2\left(\pm\sqrt{-\frac{p_2+3a_2^2k^2p_1\alpha(p_1\alpha^2-4\beta)}{24a_2^2k^4\alpha}}(\xi+\xi_0)\right)\right]}, \tag{35}$$

$$v = \frac{p_2 + 6a_2^2k(p_1\alpha^2 - 2\beta)f(t) + 3a_2^2\alpha f^2(t) - 2a_2^2k\alpha(yf'(t) + g'(t))}{6a_2^2k^2\alpha^2}$$
$$- \frac{4(p_2 + 3a_2^2k^2p_1\alpha(p_1\alpha^2 - 4\beta))f(t)\coth^2\left(\pm\sqrt{-\frac{p_2+3a_2^2k^2p_1\alpha(p_1\alpha^2-4\beta)}{24a_2^2k^4\alpha}}(\xi+\xi_0)\right)}{3a_2^2k^3\alpha(p_1\alpha^2-2\beta)\left[3+\coth^2\left(\pm\sqrt{-\frac{p_2+3a_2^2k^2p_1\alpha(p_1\alpha^2-4\beta)}{24a_2^2k^4\alpha}}(\xi+\xi_0)\right)\right]}, \tag{36}$$

where $\xi = kx + yf(t) + g(t)$.

If $(p_2 + 3a_2^2k^2p_1\alpha(p_1\alpha^2 - 4\beta))/\alpha > 0$ and $a_2(p_1\alpha^2 - 2\beta) < 0$, then from (12) and (13) we obtain trigonometric function solutions of (1) and (2):

$$u = \frac{kp_1\alpha - f(t)}{k\alpha} + \frac{4(p_2 + 3a_2^2k^2p_1\alpha(p_1\alpha^2 - 4\beta))\tan^2\left(\pm\sqrt{\frac{p_2+3a_2^2k^2p_1\alpha(p_1\alpha^2-4\beta)}{24a_2^2k^4\alpha}}(\xi+\xi_0)\right)}{3a_2^2k^2\alpha(p_1\alpha^2-2\beta)\left[3-\tan^2\left(\pm\sqrt{\frac{p_2+3a_2^2k^2p_1\alpha(p_1\alpha^2-4\beta)}{24a_2^2k^4\alpha}}(\xi+\xi_0)\right)\right]}, \tag{37}$$

$$v = \frac{p_2 + 6a_2^2k(p_1\alpha^2 - 2\beta)f(t) + 3a_2^2\alpha f^2(t) - 2a_2^2k\alpha(yf'(t) + g'(t))}{6a_2^2k^2\alpha^2}$$
$$+ \frac{4(p_2 + 3a_2^2k^2p_1\alpha(p_1\alpha^2 - 4\beta))f(t)\tan^2\left(\pm\sqrt{\frac{p_2+3a_2^2k^2p_1\alpha(p_1\alpha^2-4\beta)}{24a_2^2k^4\alpha}}(\xi+\xi_0)\right)}{3a_2^2k^3\alpha(p_1\alpha^2-2\beta)\left[3-\tan^2\left(\pm\sqrt{\frac{p_2+3a_2^2k^2p_1\alpha(p_1\alpha^2-4\beta)}{24a_2^2k^4\alpha}}(\xi+\xi_0)\right)\right]}, \tag{38}$$

where $\xi = kx + yf(t) + g(t)$;

$$u = \frac{kp_1\alpha - f(t)}{k\alpha} + \frac{4(p_2 + 3a_2^2k^2p_1\alpha(p_1\alpha^2 - 4\beta))\cot^2\left(\pm\sqrt{\frac{p_2+3a_2^2k^2p_1\alpha(p_1\alpha^2-4\beta)}{24a_2^2k^4\alpha}}(\xi+\xi_0)\right)}{3a_2^2k^2\alpha(p_1\alpha^2-2\beta)\left[3-\cot^2\left(\pm\sqrt{\frac{p_2+3a_2^2k^2p_1\alpha(p_1\alpha^2-4\beta)}{24a_2^2k^4\alpha}}(\xi+\xi_0)\right)\right]}, \tag{39}$$

$$v=\frac{p_2+6a_2^2k(p_1\alpha^2-2\beta)f(t)+3a_2^2\alpha f^2(t)-2a_2^2k\alpha(yf'(t)+g'(t))}{6a_2^2k^2\alpha^2}+\frac{4(p_2+3a_2^2k^2p_1\alpha(p_1\alpha^2-4\beta))f(t)\cot^2\left(\pm\sqrt{\frac{p_2+3a_2^2k^2p_1\alpha(p_1\alpha^2-4\beta)}{24a_2^2k^4\alpha}}(\xi+\xi_0)\right)}{3a_2^2k^3\alpha(p_1\alpha^2-2\beta)\left[3-\cot^2\left(\pm\sqrt{\frac{p_2+3a_2^2k^2p_1\alpha(p_1\alpha^2-4\beta)}{24a_2^2k^4\alpha}}(\xi+\xi_0)\right)\right]},\tag{40}$$

where $\xi=kx+yf(t)+g(t)$.

When setting

$$a_1=0,\quad h_0=0,\quad 2a_2^2k^2(p_1\alpha^2-2\beta)^2-\alpha(p_2+3a_2^2k^2p_1\alpha(p_1\alpha^2-4\beta))=0$$

in Case 1, we can verify that $h_1=h_3=h_5=0$, and h_0, h_2, h_4 and h_6 satisfy the conditions $h_0=0$, $h_6=\frac{h_4^2}{4h_2}$ in III.

If $p_2+3a_2^2k^2p_1\alpha(p_1\alpha^2-4\beta)>0$ and $a_2(p_1\alpha^2-2\beta)<0$, then from (14) and (15) we obtain soliton-like solutions of (1) and (2):

$$u=\frac{kp_1\alpha-f(t)}{k\alpha}-\frac{(p_2+3a_2^2k^2p_1\alpha(p_1\alpha^2-4\beta))\left[1+\tanh\left(\pm\sqrt{\frac{p_2+3a_2^2k^2p_1\alpha(p_1\alpha^2-4\beta)}{8a_2^2k^4\alpha}}(\xi+\xi_0)\right)\right]}{2a_2^2k^2\alpha(p_1\alpha^2-2\beta)},\tag{41}$$

$$v=\frac{p_2+6a_2^2k(p_1\alpha^2-2\beta)f(t)+3a_2^2\alpha f^2(t)-2a_2^2k\alpha(yf'(t)+g'(t))}{6a_2^2k^2\alpha^2}-\frac{(p_2+3a_2^2k^2p_1\alpha(p_1\alpha^2-4\beta))f(t)\left[1+\tanh\left(\pm\sqrt{\frac{p_2+3a_2^2k^2p_1\alpha(p_1\alpha^2-4\beta)}{8a_2^2k^4\alpha}}(\xi+\xi_0)\right)\right]}{2a_2^2k^3\alpha(p_1\alpha^2-2\beta)},\tag{42}$$

where $\xi=kx+yf(t)+g(t)$;

$$u=\frac{kp_1\alpha-f(t)}{k\alpha}-\frac{(p_2+3a_2^2k^2p_1\alpha(p_1\alpha^2-4\beta))\left[1+\coth\left(\pm\sqrt{\frac{p_2+3a_2^2k^2p_1\alpha(p_1\alpha^2-4\beta)}{8a_2^2k^4\alpha}}(\xi+\xi_0)\right)\right]}{2a_2^2k^2\alpha(p_1\alpha^2-2\beta)},\tag{43}$$

$$v=\frac{p_2+6a_2^2k(p_1\alpha^2-2\beta)f(t)+3a_2^2\alpha f^2(t)-2a_2^2k\alpha(yf'(t)+g'(t))}{6a_2^2k^2\alpha^2}-\frac{(p_2+3a_2^2k^2p_1\alpha(p_1\alpha^2-4\beta))f(t)\left[1+\coth\left(\pm\sqrt{\frac{p_2+3a_2^2k^2p_1\alpha(p_1\alpha^2-4\beta)}{8a_2^2k^4\alpha}}(\xi+\xi_0)\right)\right]}{2a_2^2k^3\alpha(p_1\alpha^2-2\beta)},\tag{44}$$

where $\xi=kx+yf(t)+g(t)$.

It should be noted that if setting $p_1=a_0+l/\alpha$, $p_2=-a_2^2k^2(9l^2\alpha-6b_0\alpha^2+6a_0l\alpha^2-12l\beta+2\alpha\lambda)$, $f(t)=kl$ and $g(t)=-k\lambda t$ in (33)–(44), then all solutions (32)–(37) in [49] can be recovered. Here we would like to point out that each term $[6(12\beta a_0+6l^2+2\lambda-3\alpha^2a_0^2-6\alpha b_0)]^{1/2}$ of the solutions (34) and (35) obtained by Li and Zhang in [49] should be corrected as $[-6(12\beta a_0+6l^2+2\lambda-3\alpha^2a_0^2-6\alpha b_0)]^{1/2}$. Also the solutions (36) and (37) in [49] should be, respectively, expressed as:

$$u_5=a_0-\frac{(12\beta a_0+6l^2+2\lambda-3\alpha^2a_0^2-6\alpha b_0)\left(1+\tanh\left[\pm\frac{\sqrt{-2(12\beta a_0+6l^2+2\lambda-3\alpha^2a_0^2-6\alpha b_0)}}{4k}(\eta+\eta_0)\right]\right)}{2(-\alpha l-\alpha^2a_0+2\beta)},$$

$$v_5 = b_0 - \frac{l(12\beta a_0 + 6l^2 + 2\lambda - 3\alpha^2 a_0^2 - 6\alpha b_0)\left(1 + \tanh\left[\pm\frac{\sqrt{-2(12\beta a_0+6l^2+2\lambda-3\alpha^2 a_0^2-6\alpha b_0)}}{4k}(\eta + \eta_0)\right]\right)}{2(-\alpha l - \alpha^2 a_0 + 2\beta)},$$

and

$$u_6 = a_0 - \frac{(12\beta a_0 + 6l^2 + 2\lambda - 3\alpha^2 a_0^2 - 6\alpha b_0)\left(1 + \coth\left[\pm\frac{\sqrt{-2(12\beta a_0+6l^2+2\lambda-3\alpha^2 a_0^2-6\alpha b_0)}}{4k}(\eta + \eta_0)\right]\right)}{2(-\alpha l - \alpha^2 a_0 + 2\beta)},$$

$$v_6 = b_0 - \frac{l(12\beta a_0 + 6l^2 + 2\lambda - 3\alpha^2 a_0^2 - 6\alpha b_0)\left(1 + \coth\left[\pm\frac{\sqrt{-2(12\beta a_0+6l^2+2\lambda-3\alpha^2 a_0^2-6\alpha b_0)}}{4k}(\eta + \eta_0)\right]\right)}{2(-\alpha l - \alpha^2 a_0 + 2\beta)}.$$

We next use Case 2 to obtain exact solutions of (1) and (2). When setting

$$h_0 = \frac{8h_2^2}{27h_4}, \quad h_6 = \frac{h_4^2}{4h_2}, \quad h_2 > 0, \quad h_4 < 0$$

in Case 2, then from (12) and (13) we obtain trigonometric function solutions of (1) and (2):

$$\begin{aligned} u = {} & \frac{2k\beta - \alpha f(t)}{k\alpha^2} \mp \frac{8k\sqrt{h_2}\tan^2\left(\pm\sqrt{\frac{h_2}{3}}(\xi+\xi_0)\right)}{3\alpha\left[3 - \tan^2\left(\pm\sqrt{\frac{h_2}{3}}(\xi+\xi_0)\right)\right]} \\ & + \frac{k\sqrt{3h_2}\cot^2\left(\pm\sqrt{\frac{h_2}{3}}(\xi+\xi_0)\right)\sin\left(\pm 2\sqrt{\frac{h_2}{3}}(\xi+\xi_0)\right)\left[3 - \tan^2\left(\pm\sqrt{\frac{h_2}{3}}(\xi+\xi_0)\right)\right]}{\alpha\left[1 + 2\cos\left(\pm 2\sqrt{\frac{h_2}{3}}(\xi+\xi_0)\right)\right]}, \end{aligned} \tag{45}$$

$$\begin{aligned} v = {} & \frac{12\beta^2k^2 - 4h_2k^4\alpha^2 + 3a_2^2f^2(t) - 2k\alpha^2(yf'(t) + g'(t))}{6k^2\alpha^3} \mp \frac{8\sqrt{h_2}f(t)\tan^2\left(\pm\sqrt{\frac{h_2}{3}}(\xi+\xi_0)\right)}{3\alpha\left[3 - \tan^2\left(\pm\sqrt{\frac{h_2}{3}}(\xi+\xi_0)\right)\right]} \\ & + \frac{\sqrt{3h_2}f(t)\cot^2\left(\pm\sqrt{\frac{h_2}{3}}(\xi+\xi_0)\right)\sin\left(\pm 2\sqrt{\frac{h_2}{3}}(\xi+\xi_0)\right)\left[3 - \tan^2\left(\pm\sqrt{\frac{h_2}{3}}(\xi+\xi_0)\right)\right]}{\alpha\left[1 + 2\cos\left(\pm 2\sqrt{\frac{h_2}{3}}(\xi+\xi_0)\right)\right]}, \end{aligned} \tag{46}$$

where $\xi = kx + yf(t) + g(t)$;

$$u = \frac{2k\beta - \alpha f(t)}{k\alpha^2} \mp \frac{8k\sqrt{h_2}\cot^2\left(\pm\sqrt{\frac{h_2}{3}}(\xi+\xi_0)\right)}{3\alpha\left[3 - \cot^2\left(\pm\sqrt{\frac{h_2}{3}}(\xi+\xi_0)\right)\right]} + \frac{2k\sqrt{3h_2}\tan\left(\pm\sqrt{\frac{h_2}{3}}(\xi+\xi_0)\right)}{\alpha\left[-1 + 2\cos\left(\pm 2\sqrt{\frac{h_2}{3}}(\xi+\xi_0)\right)\right]}, \tag{47}$$

$$v=\frac{12\beta^2k^2-4h_2k^4\alpha^2+3a_2^2f^2(t)-2k\alpha^2(yf'(t)+g'(t))}{6k^2\alpha^3}$$
$$\mp\frac{8\sqrt{h_2}f(t)\cot^2\left(\pm\sqrt{\frac{h_2}{3}}(\xi+\xi_0)\right)}{3\alpha\left[3-\cot^2\left(\pm\sqrt{\frac{h_2}{3}}(\xi+\xi_0)\right)\right]}+\frac{2k\sqrt{3h_2}f(t)\tan\left(\pm\sqrt{\frac{h_2}{3}}(\xi+\xi_0)\right)}{\alpha\left[-1+2\cos\left(\pm2\sqrt{\frac{h_2}{3}}(\xi+\xi_0)\right)\right]}, \tag{48}$$

where $\xi=kx+yf(t)+g(t)$.

When setting

$$h_0=0,\quad h_6=\frac{h_4^2}{4h_2},\quad h_2>0,\quad h_4<0$$

in Case 2, then from (14) and (15) we get soliton-like solutions of (1) and (2):

$$u=\frac{2k\beta-\alpha f(t)}{k\alpha^2}\pm\frac{k\sqrt{h_2}\left[1+\tanh\left(\pm\sqrt{h_2}(\xi+\xi_0)\right)\right]}{\alpha}+\frac{k\sqrt{h_2}\operatorname{sech}^2\left(\pm\sqrt{h_2}(\xi+\xi_0)\right)}{\alpha\left[1+\tanh\left(\pm\sqrt{h_2}(\xi+\xi_0)\right)\right]}, \tag{49}$$

$$v=\frac{12\beta^2k^2-4h_2k^4\alpha^2+3a_2^2f^2(t)-2k\alpha^2(yf'(t)+g'(t))}{6k^2\alpha^3}$$
$$\pm\frac{\sqrt{h_2}f(t)\left[1+\tanh\left(\pm\sqrt{h_2}(\xi+\xi_0)\right)\right]}{\alpha}+\frac{\sqrt{h_2}f(t)\operatorname{sech}^2\left(\pm\sqrt{h_2}(\xi+\xi_0)\right)}{\alpha\left[1+\tanh\left(\pm\sqrt{h_2}(\xi+\xi_0)\right)\right]}, \tag{50}$$

where $\xi=kx+yf(t)+g(t)$;

$$u=\frac{2k\beta-\alpha f(t)}{k\alpha^2}\pm\frac{k\sqrt{h_2}\left[1+\coth\left(\pm\sqrt{h_2}(\xi+\xi_0)\right)\right]}{\alpha}-\frac{k\sqrt{h_2}\operatorname{csch}^2\left(\pm\sqrt{h_2}(\xi+\xi_0)\right)}{\alpha\left[1+\coth\left(\pm\sqrt{h_2}(\xi+\xi_0)\right)\right]}, \tag{51}$$

$$v=\frac{12\beta^2k^2-4h_2k^4\alpha^2+3a_2^2f^2(t)-2k\alpha^2(yf'(t)+g'(t))}{6k^2\alpha^3}$$
$$\pm\frac{\sqrt{h_2}f(t)\left[1+\coth\left(\pm\sqrt{h_2}(\xi+\xi_0)\right)\right]}{\alpha}-\frac{\sqrt{h_2}f(t)\operatorname{csch}^2\left(\pm\sqrt{h_2}(\xi+\xi_0)\right)}{\alpha\left[1+\coth\left(\pm\sqrt{h_2}(\xi+\xi_0)\right)\right]}, \tag{52}$$

where $\xi=kx+yf(t)+g(t)$.

All solutions obtained above can not be obtained by the methods of [46–49]. To the best of our knowledge, they are new and have not be found in [21, 31, 33, 51–55]. It shows that our method is more powerful than the methods of [46–49]. The method proposed in this paper can be applied to other NLEEs, such as the (3+1)-dimensional Kadomtsev-Petviashvili equation [20], (2+1)-dimensional Broer-Kaup-Kupershmidt equations [25], dispersive long wave equations [20], breaking solition equations [32], Nizhnik-Novikov-Vesselov equations [37], KdV equation with variable coefficients [44].

Remark 2. All solutions obtained in this paper have been checked with *Mathematica* by putting them back into the original equations (1) and (2).

4. Conclusion

In this paper, we have proposed a generalized transformation method to seek more general exact solutions of NLEEs. Applying this method to the (2+1)-dimensional KD equations, we have obtained new and more general exact solutions including soliton-like solutions and trigonometric function solutions, from which all solutions obtained in [49] can be recovered as special cases. Compared with the methods of [46–

49], our method is more powerful in searching for exact solutions of NLEEs. The arbitrary functions in the obtained solutions imply that these solutions have rich local structures. It may be important to explain some physical phenomena. This method can be applied to other NLEEs in mathematical physics.

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